MATHEMATICS

# A SELF-DUAL SYSTEM OF AXIOMS FOR BOOLEAN ALGEBRA

BY

P. J. VAN ALBADA

(Communicated by Prof. H. FREUDENTHAL at the meeting of February 29, 1964)

## 1. *Introduction*

Since E. V. HUNTINGTON wrote his "Sets of independent postulates for the algebra of logic" [1]) several other approaches to Boolean algebra have been developed. Boolean algebras were treated as special lattices [2]) as well as as special rings [3]). However one of his sets of postulates survived in litterature. His self-dual set of axioms is the only system given in J. E. WHITESITT's book "Boolean algebra and its applications" [4]) and the first one given in R. L. GOODSTEIN's "Boolean Algebra" [5]). In this system a Boolean algebra is defined as a set with two operations (referred to in this article as addition and multiplication) which are commutative, left hand distributive with respect to each other, and both possess a righthand identity. If these identities are unique the set of equations $a+x=1$, $ax=0$ has at least one solution for every $a$.

The preassumption "if these identities are unique", which was omitted as well in Whitesitt's as in Goodstein's list of axioms is essential for the independence of the postulates. With this preassumption Huntington was able to prove the independence of his axioms simply by listing nine examples of systems of two elements only. The simplicity of this proof, together with the symmetry of his system just constitute its beauty. There is, however, no logical necessity to state the uniqueness of the identities in the axiom of complementation. If this uniqueness is not preassumed the commutativity of one operation can quite easily be derived from the remaining axioms. So from $a+b=b+a$, $a+0=a$, $a1=a$, $a(b+c)=ab+ac$, $a+bc=(a+b)(a+c)$ and for all $a$ there exists an element $a'$ such that $a+a'=1$, $aa'=0$ it follows:

---

[1]) 1904. Trans. Amer. Math. Soc. **5**, 208–309.

[2]) E.g. G. BIRKHOFF, Lattice Theory 2nd ed. New York. Amer. Math. Society Colloquium Publications, Vol. XXV.

[3]) E.g. M. H. STONE, Subsumption of Boolean algebras under the theory of rings. 1935. Proc. Nat. Acad. Sci. **21**, 103–5.

[4]) Addison–Wesley. 1960.

[5]) Pergamon Press, 1963.

$$0+a=a+0=a,$$
$$a0=a0+0=a0+aa'=a(0+a')=aa'=0,$$
$$a+a=(a+a)1=(a+a)(a+a')=a+aa'=a+0=a,$$
$$1a=(a+a')(a+0)=a+a'0=a+0=a,$$
$$a+1=(a+1)1=(a+1)(a+a')=a+1a'=a+a'=1,$$
$$0a=0a+0=0a+01=0(a+1)=0\cdot 1=0,$$
$$a+ba=(a+b)(a+a)=(a+b)a=(a+b)(a+0)=a+b0=a+0=a,$$
$$a+ab=(a+a)(a+b)=(a+0)(a+b)=a+0b=a+0=a,$$
$$ab=(a+ba)(b+ba)=(ba+a)(ba+b)=ba+ab=ab+ba=$$
$$=(ab+b)(ab+a)=ba.$$

This study will show that the other axiom of commutativity can be dropped too, if only we accept its consequences that at least one of the identities is a twosided identity and that one of the operations is both lefthand and righthand distributive with respect to the other.

## 2. *The self dual system of axioms*

Definition. A B.A. is a set, closed under two binary operations (addition and multiplication) with the following properties:

A 1. There exists a onesided identity with respect to addition.

A 2. There exists a onesided identity with respect to multiplication.

A 3. Multiplication is onesided distributive over addition.

A 4. Addition is onesided distributive over multiplication.

A 5. If both operations possess a onesided identity, then one of these identities is even twosided.

A 6. If both operations are onesided distributive over the other then one of them is even twosided distributive over the other.

A 7. If onesided identities exist for both operations we can select a onesided identity with respect to multiplication (1) and a onesided identity with respect to addition (0), so that for any $a$ the set of equations $a+x=1$, $ax=0$ has at least one solution.

Since the above axioms don't mention which of the operations is twosided distributive and whether the other is lefthand or righthand distributive, which of them has a twosided identity and whether the other identity is a lefthand or a righthand one, the above system can be realised in $2^4=16$ different ways.

To prove that a B.A. is a Boolean algebra we have to discuss 8 cases only, because of the duality of the set of axioms. So, without loss of generality, we may assume that it is the operation of multiplication which is twosided distributive over addition. Then in all 8 cases we will have the axioms $a(b+c)=ab+ac$; $(b+c)a=ba+ca$; $aa'=0$, $a+a'=1$ and, in addition to them, the following other axioms:

Case 1. $a+bc=(a+b)(a+c)$; $a+0=0+a=a$; $a1=a$.

Case 2. $a+bc=(a+b)(a+c)$; $a+0=0+a=a$; $1a=a$.

Case 3. $a+bc=(a+b)(a+c)$; $a1=1a=a$; $a+0=a$.

Case 4. $a+bc=(a+b)(a+c)$; $a1=1a=a$; $0+a=a$.

Case 5. $bc+a=(b+a)(c+a)$; $a+0=0+a=a$; $a\cdot 1=a$.

Case 6. $bc+a=(b+a)(c+a)$; $a+0=0+a=a$; $1\cdot a=a$.

Case 7. $bc+a=(b+a)(c+a)$; $a\cdot 1=1\cdot a=a$; $a+0=a$.

Case 8. $bc+a=(b+a)(c+a)$; $a\cdot 1=1\cdot a=a$; $0+a=a$.

## 3. *Proofs that each of these systems leads to Boolean algebra*

Case 1. Exactly as was done in section 2 we prove that $a\cdot 0=0$; $a+a=a$; $1\cdot a=a$, $a+1=1$; $0\cdot a=0$.

Further: $a(a+b)=(a+0)(a+b)=a+0b=a+0=a$.
$a(b+a)=ab+aa=ab+a=ab+a\cdot 1=a(b+1)=a\cdot 1=a$.
$(a+b)a=(a+b)(a+0)=a+b\cdot 0=a+0=a$.
$(b+a)a=ba+aa=ba+a=ba+1\cdot a=(b+1)a=1\cdot a=a$.

Therefore $a+b=(b+a)a+(b+a)b=(b+a)(a+b)=b(a+b)+a(a+b)=$
$=b+a$.

Addition is commutative and so, by section 2, is multiplication. We have proved the validity of all Huntington's axioms.

Case 2. By $0\cdot a'=0+0\cdot a'=aa'+0\cdot a'=(a+0)a'=aa'=0$ and $a\cdot 1=$ $=(a+0)(a+a')=a+0\cdot a'=a+0=a$ we are led to case 1.

Case 3. This case can be reduced to case 1 as follows:
$a+1=(a+1)\cdot 1=(a+1)(a+a')=a+1\cdot a'=a+a'=1$,
$0+a=aa'+a\cdot 1=a(a'+1)=a\cdot 1=a$.

Case 4. We derive (writing $a''$ for $(a')'$):
$a+1=1\cdot(a+1)=(a+a')(a+1)=a+a'\cdot 1=a+a'=1$.
$aa''=0+aa''=aa'+aa''=a(a'+a'')=a\cdot 1=a$.
$a+a''=aa''+1\cdot a''=(a+1)a''=1\cdot a''=a''$.
$a'a'=0+a'a'=aa'+a'a'=(a+a')a'=1\cdot a'=a'$.
$a'+0=a'a'+a'a''=a'(a'+a'')=a'\cdot 1=a'$.
$a+0=a+a'a''=(a+a')(a+a'')=1\cdot a''=a''$.
$(a+0)+0=a''+0=a''=a+0$.
$aa+0=aa+aa'=a(a+a')=a\cdot 1=a$.
$a+0=(aa+0)+0=aa+0=a$.

We are back to case 1.

Case 5. Commutativity of both operations can be shown as before as soon as $a(a+b)=a(b+a)=(a+b)a=(b+a)a=a$ has been proved.

This is done as follows:

$aa = aa + 0 = aa + aa' = a(a + a') = a1 = a.$
$a0 = 0 + a0 = aa' + a0 = a(a' + 0) = aa' = 0.$
$a + a'' = (a + a'')1 = (a + a'')(a' + a'') = aa' + a'' = 0 + a'' = a''.$
$a'a = a'a + 0 = a'a + a'a'' = a'(a + a'') = a'a'' = 0.$
$1 \cdot a = (a + a')a = aa + a'a = aa + 0 = aa = a.$
$a'' = 1 \cdot a'' = (a + a')a'' = aa'' + a'a'' = aa'' + 0 = aa'' = 0 + aa'' = aa' + aa'' =$
$= a(a' + a'') = a1 = a.$
$a' + a = a' + a'' = 1.$
$0a = 0 + 0a = a'a + 0a = (a' + 0)a = a'a = 0.$
$a(b + a) = (0 + a)(b + a) = 0b + a = 0 + a = 0.$
$(b + a)a = (b + a)(0 + a) = b0 + a = 0 + a = 0.$
$1 + a = 1(1 + a) = (a' + a)(1 + a) = a' + a = 1.$
$a(a + b) = aa + ab = a1 + ab = a(1 + b) = a1 = a.$
$(a + b)a = aa + ba = 1a + ba = (1 + b)a = 1a = a.$

Case 6 is reduced to case 5 by
$0 \cdot a' = 0 + 0a' = aa' + 0a' = (a + 0)a' = aa' = 0.$
$aa = 0 + aa = aa' + aa = a(a' + a) = (0 + a)(a' + a) = 0a' + a = 0 + a = a.$
$a1 = a(a + a') = aa + aa' = a + 0 = a.$

Case 7 is reduced to case 5 by
$aa = aa + 0 = aa + aa' = a(a + a') = a1 = a.$
$0 + a' = aa' + a'a' = (a + a')a' = 1a' = a'.$
$a'' = 0 + a'' = 0 + 1a'' = 0 + (a + a')a'' = 0 + (aa'' + a'a'') = 0 + (aa'' + 0) =$
$= 0 + aa'' = aa' + aa'' = a(a' + a'') = a1 = a.$
$0 + a = 0 + a'' = a'' = a.$

Case 8 is reduced to case 5 by
$1 + a' = 1(1 + a') = (a + a')(1 + a') = a1 + a' = a + a' = 1.$
$a + 0 = a + aa' = a1 + aa' = a(1 + a') = a1 = a.$

## 4. *Independence of the postulates*

a. The set of real numbers $x$, $0 < x \leqslant 1$ for which the operations $(+)$ and $(\cdot)$ are defined by

$$a + b = \max\,(a, b); \qquad a \cdot b = \min\,(a, b)$$

satisfies all axioms apart from A1.

b. As above, take $0 \leqslant x < 1$. All axioms but A2 are satisfied.

c. As before, take $0 \leqslant x \leqslant 1$. All axioms but A7 hold.

d. The Galois field with 2 elements satisfies all axioms but A4.

e. The dual system satisfies all axioms but A3.

f. Let $B = (a, b, c, \ldots, 0, 1)$ be a Boolean algebra. We derive another set $A$ by affixing indices $1, \ldots, n$ to the elements of $B$. From $A$ we make an algebraic system by introducing the following rules:

If in $B$ the equalities $ab = c$, $a + b = d$ hold, then in $A$: $a_i b_j = c_j$, $a_i + b_j = d_j$ will hold. Obviously all $0_i$ and all $1_j$ are lefthand identifies, righthand identities however don 't exist. If we select a pair $0_i$, $1_i$, then $a_i'$ is a complement to every $a_j$. All lefthand and righthand distributive laws are valid. Therefore A5 is the only axiom which is not satisfied.

g. As in f, this time no indices will be attached to the elements 0 and 1, which will operate as twosides identities. Let in $B$ $(a+b)c = ac + bc = p$. Compare in $A$ the elements $(a_i + b_j)c_k$ and $a_i c_k + b_j c_k$. If $p = 0$ of $p = 1$ we will have $(a_i + b_j)c_k = a_i c_k + b_j c_k$. The same will be true if $c$ has no index.

In all other cases both elements will be equal to $p_k$, because at most one of the products $a_i c_k$ and $b_j c_k$ can be equal to zero. So multiplication is righthand distributive over addition, A3 holds and so does A4, because of duality. Every $a_j$ is a complement to every $a_i$, so A7 holds.

But A6 does not hold as is seen from

$b + a' = c$, $a(b + a') = ab + aa' = ab = d$.
$a_i(b_j + a_i') = a_i c_i = d_i$.
$a_i b_j + a_i a_i' = d_j + 0 = d_j$.
and, dually, $a_i + b_j a_i' \neq (a_i + b_j)(a_i + a_i')$.